Fill Ups, True Fals of Matrices and Determinants

Fill in the Blanks

Q. 1. Let $p\lambda^4 + q\lambda^3 + r\lambda^2 + s\lambda + t = \begin{vmatrix} \lambda^2+3\lambda & \lambda-1 & \lambda+3 \\ \lambda+1 & -2\lambda & \lambda-4 \\ \lambda-3 & \lambda+4 & 3\lambda \end{vmatrix}$ **be an identity in λ , where p, q, r, s and t are constants.**

Then, the value of t is .... (1981 - 2 Marks)

Ans. t = 0

Solution. As given equation is an identity in λ, it must be true for all values of λ.

∴ For λ = 0 also. Putting λ = 0 we get $t = \begin{vmatrix} 0 & -1 & 3 \\ 1 & 0 & -4 \\ -3 & 4 & 0 \end{vmatrix} = 0$

Q. 2. The solution set of the equation $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$ is **.................. (1981 - 2 Marks)**

Ans. x = –1, 2

Solution. Given equation is, $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$

Clearly on expanding the det. we will get a quadratic equation in x.

∴ It has 2 roots. We observe that R3 becomes identical to R_1 if x = 2. thus at x = 2 ⇒ Δ = 0

∴ x = 2 is a root of given eq.

Similarly R_3 becomes identical to R_2 if x = – 1. thus at x = – 1 Δ D = 0

∴ x = – 1 is a root of given eq.

Hence equation has roots as –1 and 2.

Q. 3. A determinant is chosen at random from the set of all determinants of order 2 with elements 0 or 1 only. The probability that the value of determinant chosen is positive is .... (1982 - 2 Marks)

Ans. 3/16

Solution. With 0 and 1 as elements there are 2 × 2 × 2 × 2 = 16 determinants of order 2 × 2 out of which only $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$ are the three det whose value is +ve.

∴ Req. prob. = 3/16

Q. 4. Given that x = -9 is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$ **the other two roots are .... and .... (1983 - 2 Marks)**

Ans. 2, 7

Solution.

$$\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$$

Operating $R_1 \to R_1 + R_2 + R_3$ we get

$$\begin{vmatrix} x+9 & x+9 & x+9 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$$

$$\Rightarrow (x+9)\begin{vmatrix} 1 & 1 & 1 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$$

Operating $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$

$$\Rightarrow \quad (x+9)\begin{vmatrix} 1 & 0 & 0 \\ 2 & x-2 & 0 \\ 7 & -1 & x-7 \end{vmatrix} = 0$$

Expanding along R_1

$\Rightarrow (x + 9)(x - 2)(x - 7) = 0$

$\Rightarrow x = -9, 2, 7$

$\therefore$ Other roots are 2 and 7.

Q. 5. The system of equations

$$\begin{aligned} \lambda x + y + z &= 0 \\ -x + \lambda y + z &= 0 \\ -x - y + \lambda z &= 0 \end{aligned}$$

Will have a non-zero solution if real values of l are given by.................. (1984 - 2 Marks)

Ans. $\lambda = 0$

Solution. The given homogeneous system of equations will have non zero solution if D = 0

$$\Rightarrow \begin{vmatrix} \lambda & 1 & 1 \\ -1 & \lambda & 1 \\ -1 & -1 & \lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda(\lambda^2 + 1) - 1(-\lambda + 1) + 1(1 + \lambda) = 0 \Rightarrow \lambda^3 + 3\lambda = 0$

$\Rightarrow \lambda(\lambda^2 + 3) = 0$, but $\lambda^2 + 3 \neq 0$ for real $\lambda \Rightarrow \lambda = 0$

Q. 6. The value of the determinant $\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix}$ **is.................. (1988 - 2 Marks)**

Ans. 0

Solution.

$$\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$$

Operating $R_1 \to R_1 - R2$; $R2 \to R2 - R_3$

$$\begin{vmatrix} 0 & a-b & (a-b)(a+b+c) \\ 0 & b-c & (b-c)(a+b+c) \\ 1 & c & c^2-ab \end{vmatrix}$$

$$=(a-b(b-c)\begin{vmatrix} 0 & 1 & a+b+c \\ 0 & 1 & a+b+c \\ 1 & c & c^2-ab \end{vmatrix}=0$$

Q. 7. For positive numbers x, y and z, the numerical value of the determinant $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ is .... **(1993 - 2 Marks)**

Ans. 0

Solution. Given x, y, z and + ve numbers, then value of

$$D=\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & \frac{\log y}{\log x} & \frac{\log z}{\log x} \\ \frac{\log x}{\log y} & 1 & \frac{\log z}{\log y} \\ \frac{\log x}{\log z} & \frac{\log y}{\log z} & 1 \end{vmatrix} \quad \left(\because \log_b a = \frac{\log a}{\log b}\right)$$

Taking $\frac{1}{\log x}, \frac{1}{\log y}$, and $\frac{1}{\log z}$ common from R_1, R_2 and R_3 respectively

$$D = \frac{1}{\log x \log y \log z}\begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log z \end{vmatrix} = 0$$

True / False

Q. 1. The determinants $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ and $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ are not identically equal. (1983 - 1 Mark)

Ans. F

Solutions.

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$$

$$= \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = (-1)^2 \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

[$C_1 \Leftrightarrow C_3$ and then $C_2 \Leftrightarrow C_3$]

∴ Equal. Hence statement is F.

Q. 2. If $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$ then the two triangles with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3), and (a_1, b_1), (a_2, b_2), (a_3, b_3) must be congruent. (1985 - 1 Mark)

Ans. F

Solutions.

(i) $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$

Ar (Δ_1) = Ar (D_2)

Where Δ_1 is the area of triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3); and Δ_2 is the area of triangle with; vertices (a_1, b_1), (a_2, b_2) and (a_3, b_3). But two D's of same area may not be congruent.

∴ Given statement is false.

Subjective Questions of Matrices and Determinants

Q.1. For what value of k do the following system of equations possess a non trivial (i.e., not all zero) solution over the set of rationals Q?

x + ky + 3z = 0
3x + ky – 2z = 0
2x + 3y – 4z = 0
For that value of k, find all the solutions for the system. (1979)

Ans. $x = b,\ y = \frac{-2b}{15},\ z = \frac{2b}{5},\ b \in R$

Solution. We should have,

$$\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{vmatrix} = 0$$

$$\Rightarrow 1(-4k+6) - k(-12+4) + 3(9-2k) = 0$$

$$\Rightarrow -2k + 33 = 0 \Rightarrow k = \frac{33}{2}$$

Substituting $k = \frac{33}{2}$ and putting x = b, where b ∈ Q, we get the system as

33y + 6z = – 2b ...(1)

33y – 4z = – 6b ...(2)

3y – 4z = – 2b ...(3)

$$(1)-(2) \Rightarrow 10z = 4b \Rightarrow z = \frac{2}{5}b$$

$$(1) \Rightarrow 33y = -2b - \frac{12b}{5} = -\frac{22b}{5} \Rightarrow y = \frac{-2b}{15}$$

$$\therefore \text{The solution is } x = b,\ y = \frac{-2b}{15},\ z = \frac{2b}{5}$$

Q.2. Let a, b, c be positive and not all equal. Show that the value of the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is negative. **(1981 - 4 Marks)**

Solution. The given det, on expanding along R_1, we get

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a(bc - a^2) - b(b^2 - ac) + c(ab - c^2)$$

$= 3abc - a^3 - b^3 - c^3 = -(a^3 + b^3 + c^3 - 3abc)$

$= -(a + b + c)[a^2 + b^2 + c^2 - ab - bc - ca]$

$$= -\frac{1}{2}(a + b + c)[2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca]$$

$$= -\frac{1}{2}(a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2]$$

As $a, b, c > 0$

$\therefore a + b + c > 0$

Also $a \neq b \neq c$

$\therefore (a - b)^2 + (b - c)^2 + (c - a)^2 > 0$

Hence the given determinant is – ve.

Q.3. Without expanding a determinant at any stage, show that

$$\begin{vmatrix} x^2 + x & x + 1 & x - 2 \\ 2x^2 + 3x - 1 & 3x & 3x - 3 \\ x^2 + 2x + 3 & 2x - 1 & 2x - 1 \end{vmatrix} = xA + B,$$

where A and B are determinants of order 3 not involving x. (1982 - 5 Marks)
Solution.

$$\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = xA + B$$

$$\text{L.H.S.} = \begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$$

Operation $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$

$$\begin{vmatrix} x^2+x & x+1 & x-2 \\ x-1 & x-2 & x+1 \\ x+3 & x-2 & x+1 \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & x+1 & x-2 \\ 0 & x-2 & x+1 \\ 0 & x-2 & x+1 \end{vmatrix} + \begin{vmatrix} x & x+1 & x-2 \\ x-1 & x-2 & x+1 \\ x+3 & x-2 & x+1 \end{vmatrix}$$

$$= 0 + \begin{vmatrix} x & x+1 & x-2 \\ x-1 & x-2 & x+1 \\ x+3 & x-2 & x+1 \end{vmatrix}$$

Operating $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$

$$= \begin{vmatrix} x & x+1 & x-2 \\ -1 & -3 & 3 \\ 4 & 0 & 0 \end{vmatrix} = \begin{vmatrix} x & x & x \\ -1 & -3 & 3 \\ 4 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -1 & -3 & 3 \\ 4 & 0 & 0 \end{vmatrix}$$

$$= x\begin{vmatrix} 1 & 1 & 1 \\ -1 & -3 & 3 \\ 4 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -1 & -3 & 3 \\ 4 & 0 & 0 \end{vmatrix}$$

= xA + B = R.H. S

Hence Proved.

Q.4. Show that $\begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^xC_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^yC_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^zC_{r+2} \end{vmatrix} = \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix}$ **(1985 - 2 Marks)**

Solution. On L.H.S.= D, applying operations $C_2 \rightarrow C_2 + C_1$ and $C_3 \rightarrow C_3 + C_2$ and using ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$, we get

$$D = \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+1}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+1}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+1}C_{r+2} \end{vmatrix}$$

Operating $C_3 + C_2$ and using the same result, we get

$$D = \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix} = \text{RHS}$$

Hence proved

Q.5. Consider the system of linear equations in x, y, z :

$(\sin 3\theta)\, x - y + z = 0$

$(\cos 2\theta)\, x + 4y + 3z = 0$

$2x + 7y + 7z = 0$

Find the values of θ for which this system has nontrivial solutions. (1986 - 5 Marks)

Ans. $n\pi$ or $n\pi + (-1)^n \pi / 6,\ n \in Z$

Solution. The system will have a non-trivial solution if

$$\begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$$

Expanding along C1, we get

$\Rightarrow (28 - 21) \sin 3\theta - (-7 - 7) \cos 2\theta + 2(-3 - 4) = 0$

$\Rightarrow 7 \sin 3\theta + 14 \cos 2\theta - 14 = 0$

$\Rightarrow$ sin 3θ+ 2 cos 2θ – 2 = 0

$\Rightarrow$ 3 sin – 4 sin ^{3}θ + 2 (1 – 2 sin^{2}θ) – 2 = 0

$\Rightarrow$ 4 sin^{3}θ + 4 sin ^{2}θ – 3 sin θ = 0

$\Rightarrow$ sin θ (2 sinθ – 1) (2 sin θ + 3) = 0

sin θ = 0 or sin θ = 1/2 (sin θ = – 3/2 not possible)

$\Rightarrow$ θ = nπ or θ = nπ + (–1)^n π/6, n ∈ Z.

Q.6. Let $\Delta a = \begin{vmatrix} a-1 & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$. **Show that** $\sum_{a=1}^{n} \Delta a = c$, a constant. **(1989 - 5 Marks)**

Solution. We have

$$\Delta a = \begin{vmatrix} (a-1) & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$\text{Then } \sum_{a=1}^{n} \Delta a = \begin{vmatrix} (1-1) & n & 6 \\ (1-1)^2 & 2n^2 & 4n-2 \\ (1-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$+\begin{vmatrix} (2-1) & n & 6 \\ (2-1)^2 & 2n^2 & 4n-2 \\ (2-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix} + \ldots\ldots$$

$$+\begin{vmatrix} (n-1) & n & 6 \\ (n-1)^2 & 2n^2 & 4n-2 \\ (n-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \begin{vmatrix} 1+2+3+\ldots+(n-1) & n & 6 \\ 1^2+2^2+3^2+\ldots\ldots+(n-1)^2 & 2n^2 & 4n-2 \\ 1^3+2^3+3^3+\ldots+(n-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$=\begin{vmatrix} \frac{n(n-1)}{2} & n & 6 \\ \frac{n(n-1)(2n-1)}{6} & 2n^2 & 4n-2 \\ \left(\frac{n(n-1)}{2}\right)^2 & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$=\frac{n^2(n-1)}{12}\begin{vmatrix} 6 & 1 & 6 \\ 2(2n-1) & 2n & 2(2n-1) \\ 3n(n-1) & 3n^2 & 3n(n-1) \end{vmatrix}$$

(Taking $\frac{n(n-1)}{12}$ Common from C_1 and n from C_2)

= 0 (as C_1 and C_3 are identical)

Thus, $\sum_{a=1}^{n} \Delta a = 0 \Rightarrow \sum_{a=1}^{n} \Delta a = c$ (a constant) where $c = 0$

Q.7. Let the three digit numbers A 28, 3B9, and 62 C, where A, B, and C are integers between 0 and 9, be divisible by a fixed integer k. Show that the determinant $\begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$ is divisible by k. (1990 - 4 Marks)

Solution. Given that A, B, C are integers between 0 and 9 and the three digit numbers A28, 3B9 and 62C are divisible by a fixed integer k.

Now, $D=\begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$

On operating $R_2 \to R_2 + 10\,R_3 + 100\,R_1$, we get

$$=\begin{vmatrix} A & 3 & 6 \\ A28 & 3B9 & 62C \\ 2 & B & 2 \end{vmatrix} = \begin{vmatrix} A & 3 & 6 \\ kn_1 & kn_2 & kn_3 \\ 2 & B & 2 \end{vmatrix}$$

[As per question A28, 3B9 and 62C are divisible by k, therefore,

$A28 = kn_1$
$3B9 = kn_2$
$62C = kn_3]$

$$= k\begin{vmatrix} A & 3 & 6 \\ n_1 & n_2 & n_3 \\ 2 & B & 2 \end{vmatrix} = k \times \text{some integral value.}$$

⇒ D is divisible by k.

Q.8. If a ≠ p, b ≠ q, c ≠ r and $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0.$ **Then find the value of** $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c}$ **(1991 - 4 Marks)**

Solution.

Consider $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$

Operating $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$ we get

$$\begin{vmatrix} p-a & -(q-b) & c \\ 0 & q-b & c-r \\ a & b & r \end{vmatrix} = 0$$

Taking (p – q), (q – b) and (r – c) common from C_1, C_2 and C_3 resp, we get

$$\Rightarrow (p-a)(q-b)(r-c)\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ \frac{a}{p-a} & \frac{b}{q-b} & \frac{r}{r-c} \end{vmatrix} = 0$$

$$\Rightarrow (p-a)(q-b)(r-c)\left[1\left(\frac{r}{r-c} + \frac{b}{q-b}\right) + \frac{a}{p-a}\right] = 0$$

As given that p ≠ a, q ≠ b, r ≠ c

$$\therefore \quad \frac{r}{r-c}+\frac{b}{q-b}+\frac{a}{p-a}=0$$

$$\Rightarrow \quad \frac{r}{r-c}+\frac{q-(q-b)}{q-b}+\frac{p-(p-a)}{p-a}=0$$

$$\Rightarrow \quad \frac{r}{r-c}+\frac{q}{q-b}-1+\frac{p}{p-a}-1=0$$

$$\Rightarrow \quad \frac{p}{p-a}+\frac{q}{q-b}+\frac{r}{r-c}=2$$

Q.9. For a fixed positive integer n, if $D=\begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$ **then show that** $\left[\frac{D}{(n!)^3}-4\right]$ **is divisible by n.** **(1992 - 4 Marks)**

Solution.

$$D=\begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$$

$$=n!\,(n+1)!\,(n+2)!\begin{vmatrix} 1 & n+1 & (n+2)(n+1) \\ 1 & n+2 & (n+3)(n+2) \\ 1 & n+3 & (n+4)(n+3) \end{vmatrix}$$

Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$, we get

$$D=(n!)^3\,(n+1)^2\,(n+2)\begin{vmatrix} 1 & n+1 & n^2+3n+2 \\ 0 & 1 & 2n+4 \\ 0 & 1 & 2n+6 \end{vmatrix}$$

Operating $R_3 \to R_3 - R_2$

$$D = (n!)^3 (n+1)^2 (n+2) \begin{vmatrix} 1 & n+1 & n^2+3n+2 \\ 0 & 1 & 2n+4 \\ 0 & 0 & 2 \end{vmatrix}$$

$$= (n!)^3 (n+1)^2 (n+2)\, 1\, [2]$$

$$\Rightarrow \frac{D}{(n!)^3} = 2\ (n+1)^2 (n+2)$$

$$\Rightarrow \frac{D}{(n!)^3} - 4 = 2\,(n^3 + 4n^2 + 5n + 2) - 4$$

$$= 2\,(n^3 + 4n^2 + 5n) = 2n\,(n^2 + 4n + 5)$$

$$\Rightarrow \frac{D}{(n!)^3} - 4 \text{ is divisible by } n.$$

Q.10. Let λ and α be real. Find the set of all values of λ for which the system of linear equations (1993 - 5 Marks)
λx + (sin α) y+ (cos α) z = 0 , x + (cos α) y + (sin α) z = 0 , - x + (sin α) y - (cos α) z = 0 has a non-trivial solution. For λ = 1, find all values of α.

Solution. Given that λ, α ∈ R and system of linear equations

λx + (sin α) y + (cos α)z = 0

x + (cos α) y + (sin α) z = 0

– x (sin α) y – (cos α) z = 0

has a non trivial solution, therefore D = 0

$$\Rightarrow \begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

⇒ λ (– $\cos^2$ α – $\sin^2$α) – sin α (– cos α + sin α) + cos α(sin α + cos α) = 0

⇒ – λ + sin a cos α – $\sin^2$ α + sin α cosα + $\cos^2$α = 0

⇒ λ = $\cos^2$α – $\sin^2$α + 2 sin a cos α

⇒ λ = cos 2α + sin 2α

For l = 1, cos 2α + sin 2α = 1

$$\frac{1}{\sqrt{2}}\cos 2\alpha + \frac{1}{\sqrt{2}}\sin 2\alpha = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \cos 2\alpha \cos \pi/4 + \sin 2\alpha \sin \pi/4 = \frac{1}{\sqrt{2}}$$

$\Rightarrow \cos(2\alpha - \pi/4) = \cos p/4$

$\Rightarrow 2\alpha - \pi/4 = 2n\pi \pm \pi/4 \Rightarrow 2a = 2\pi r + \pi/4 + \pi/4; 2n\pi - \pi/4 + \pi/4$

$\Rightarrow \alpha = n\pi + p/4$ or $n\pi$

Q.11. For all values of A, B, C and P, Q, R show that (1994 - 4 Marks)

$$\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = 0$$

Solution. L.H.S.

$$= \begin{vmatrix} \cos A\cos P + \sin A \sin P & \cos(A-Q) & \cos(A-R) \\ \cos B\cos P + \sin B \sin P & \cos(B-Q) & \cos(B-R) \\ \cos C\cos P + \sin C \sin P & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

$$= \cos P\begin{vmatrix} \cos A & \cos(A-Q) & \cos(A-R) \\ \cos B & \cos(B-Q) & \cos(B-R) \\ \cos C & \cos(C-Q) & \cos(C-R) \end{vmatrix} + \sin P\begin{vmatrix} \sin A & \cos(A-Q) & \cos(A-R) \\ \sin B & \cos(B-Q) & \cos(B-R) \\ \sin C & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

Operating; $C_2 \to C_2 - C_1 (\cos Q)$; $C_3 \to C_3 - C_1 (\cos R)$ on first determinant and $C_2 \to C_2 - (\sin Q) C_1$ and $C_3 \to C_3 - (\sin R)C_1$ on second determinant, we get

$$= \cos P\begin{vmatrix} \cos A & \sin A\sin Q & \sin A\sin R \\ \cos B & \sin B\sin Q & \sin B\sin R \\ \cos C & \sin C\sin Q & \sin C\sin R \end{vmatrix} + \sin P\begin{vmatrix} \sin A & \cos A\cos Q & \cos A\cos R \\ \sin B & \cos B\cos Q & \cos B\cos R \\ \sin C & \cos C\cos Q & \cos C\cos R \end{vmatrix}$$

$$= \cos P\sin Q\sin R\begin{vmatrix} \cos A & \sin A & \sin A \\ \cos B & \sin B & \sin B \\ \cos C & \sin C & \sin C \end{vmatrix} + \sin P\cos Q\cos R\begin{vmatrix} \sin A & \cos A & \cos A \\ \sin B & \cos B & \cos B \\ \sin C & \cos C & \cos C \end{vmatrix}$$

$= 0 + 0$ [Both determinants become zero as $C_2 \equiv C_3$]

$= 0 =$ R.H.S.

Q.12. Let a > 0, d > 0. Find the value of the determinant (1996 - 5 Marks)

$$\begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{(a+d)} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{(a+2d)} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}$$

Ans. $\dfrac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$

Solution. Let us denote the given determinant by Δ Taking

$\dfrac{1}{a(a+d)(a+2d)}$ as common from

R_1, $\dfrac{1}{(a+d)(a+2d)(a+3d)}$ from R_2 and

$\dfrac{1}{(a+2d)(a+3d)(a+4d)}$ from R_3, we get

$$\Delta = \frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}\Delta_1$$

Where

$$\Delta_1 = \begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(a+3d) & a+3d & a+d \\ (a+3d)(a+4d) & a+4d & a+2d \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$, we get

$$\Delta_1 = \begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(2d) & d & d \\ (a+3d)(2d) & d & d \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$, we get

$$\Delta_1 = \begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(2d) & d & d \\ 2d^2 & 0 & 0 \end{vmatrix}$$

Expanding along R_3, we get

$$\Delta_1 = (2d^2)\begin{vmatrix} a+2d & a \\ d & d \end{vmatrix} = (2d)^2 (d)(a+2d-a) = 4d^4$$

$$\text{Thus, } \Delta = \frac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$$

Q.13. Prove that for all values of θ, **(2000 - 3 Marks)**

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix} = 0$$

Solution. $R_2 \to R_2 + R_3$,

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ 2\sin\theta\cos\frac{2\pi}{2} & 2\cos\theta\cos\frac{2\pi}{3} & 2\sin 2\theta\cos\frac{4\pi}{3} \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix} = 0$$

$$= \begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ -\sin\theta & -\cos\theta & -\sin 2\theta \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix} = 0$$

Q.14. If matrix $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$. **where a, b, c are real positive numbers, abc = 1 and $A^TA = I$, then find the value of $a^3 + b^3 + c^3$.** **(2003 - 2 Marks)**

Ans. 4

Solution. Given that ATA = I

$\Rightarrow |A^TA| = |A^T|\,|A| = |A|\,|A| = 1\ [\because |I| = I]$

$\Rightarrow |A|^2 = 1$ ....(1)

From given matrix $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$

$$|A| = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a^3 + b^3 + c^3 - 3abc$$ ....(2)

$\therefore (a^3 + b^3 + c^3 - 3abc)^2 = 1$ (From (1) and (2)]

$\Rightarrow a^3 + b^3 + c^3 - 3abc = 1$ or -1

But for a^3, b^3, c^3 using AM $\geq$ GM

We get $\frac{a^3 + b^3 + c^3}{3} \geq \sqrt[3]{a^3 b^3 c^3} \Rightarrow a^3 + b^3 + c^3 - 3abc \geq 0$

$\therefore$ We must have $a^3 + b^3 + c^3 - 3abc = 1$

$\Rightarrow a^3 + b^3 + c^3 = 1 + 3 \times 1 = 4$ [Using abc = 1]

Q.15. If M is a 3 × 3 matrix, where det M = 1 and $MM^T = I$, where 'I' is an identity matrix, prove that det (M – I) = 0. (2004 - 2 Marks)

Solution. We are given that $MM^T = I$ where M is a square matrix of order 3 and det. M = 1.

Consider det (M – I) = det $(M - M M^T)$ [Given $MM^T = I$]

= det $[M (I - M^T)]$

= (det M) (det $(I - M^T)$) [$\therefore |AB| = |A|\,|B|$]

= – (det M) (det $(M^T - I)$)

= – 1 [det $(M^T - I)$][Q det (M) = 1]

= – det (M – I)

[∴ $\det(M^T - I) = \det[(M - I)^T] = \det(M - I)$]

$\Rightarrow 2\det(M - I) = 0 \Rightarrow \det(M - I) = 0$

Hence Proved

Q.16. If $A = \begin{bmatrix} a & 1 & 0 \\ 1 & b & d \\ 1 & b & c \end{bmatrix}, B = \begin{bmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{bmatrix}, U = \begin{bmatrix} f \\ g \\ h \end{bmatrix}, V = \begin{bmatrix} a^2 \\ 0 \\ 0 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ **and AX = U has infinitely many solutions, prove that BX = V has no unique solution. Also show that if afd ≠ 0, then BX = V has no solution. (2004 - 4 Marks)**

Solution. Given that,

$$A = \begin{bmatrix} a & 1 & 0 \\ 1 & b & d \\ 1 & b & c \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, U = \begin{bmatrix} f \\ g \\ h \end{bmatrix}$$

And AX = U has infinite many solutions.

$\Rightarrow |A| = 0 = |A_1| = |A_2| = |A_3|$

Now, $|A| = 0$

$$\Rightarrow \begin{vmatrix} a & 1 & 0 \\ 1 & b & d \\ 1 & b & c \end{vmatrix} = a(bc - bd) - 1(c - d) = 0$$

$\Rightarrow (ab - 1)(c - d) = 0$

$\Rightarrow ab = 1$ or $c = d$ ....(1)

$$\text{And } |A_1| = \begin{vmatrix} f & 1 & 0 \\ g & b & d \\ h & b & c \end{vmatrix} = 0$$

$\Rightarrow f(bc - bd) - 1(gc - hd) = 0$

$\Rightarrow f\,b(c - d) = gc - hd$ ....(2)

$$|A_2| = \begin{vmatrix} a & 1 & f \\ 1 & b & g \\ 1 & b & h \end{vmatrix} = 0$$

$\Rightarrow$ a (gc – hd) – f (c – d) = 0 $\Rightarrow$ a (gc – hd) = f (c – d)

$$|A_3| = \begin{vmatrix} a & 1 & f \\ 1 & b & g \\ 1 & b & h \end{vmatrix} = 0$$

$\Rightarrow$ a(bh – bg) – 1(h – g) + f (b – b) = 0

$\Rightarrow$ ab (h – g) – (h – g) = 0

$\Rightarrow$ ab = 1 or h = g ....(3) Taking c = d

$\Rightarrow$ h = g and ab $\neq$ 1 (from (1), (2) and (3))

Now taking BX = V

$$\text{where } B = \begin{bmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{bmatrix}, V = \begin{bmatrix} a^2 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Then } |B| = \begin{vmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{vmatrix} = 0$$

[$\because$In vew of c = d and g = h, C_2 and C_3 are identical]

$\Rightarrow$ BX = V has no unique solution

$$\text{And } |B_1| = \begin{vmatrix} a^2 & 1 & 1 \\ 0 & d & c \\ 0 & g & h \end{vmatrix} = 0 \quad (\because c = d, g = h)$$

$$|B_2| = \begin{vmatrix} a & a^2 & 1 \\ 0 & 0 & c \\ f & 0 & h \end{vmatrix} = a^2cf = a^2df$$

($\because$ c = d)

$$|B_3| = \begin{vmatrix} a & 1 & a^2 \\ 0 & d & 0 \\ f & g & 0 \end{vmatrix} = a^2df$$

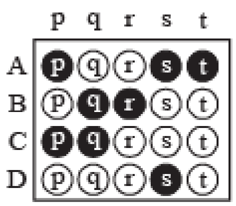
$\Rightarrow$ If adf $\neq$ 0 then | B_2 | = | B_3 | $\neq$ 0
Hence no solution exist.

Additional Questions of Matrices and Determinants

Match the Following

Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r, s and t. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in

Column II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example :
If the correct matches are A-p, s and t; B-q and r; C-p and q; and D-s then the correct darkening of bubbles will look like the given.



Q.1. Consider the lines given by

L_1 : x + 3y – 5 = 0; L_2 : 3x – ky – 1 = 0; L3 : 5x + 2y – 12 = 0
Match the Statements / Expressions in Column I with the Statements / Expressions in Column II and indicate your answer by darkening the appropriate bubbles in the 4 × 4 matrix given in the ORS.

Column I	Column II
(A) L1, L2, L3 are concurrent, if	(p) k = –9
(B) One of L1, L2, L3 is parallel to at least one of the other two, if	(q) $k = -\frac{6}{5}$
(C) L1, L2, L3 from a triangle, if	(r) k = 5/6
(D)L1, L2, L3 do not form a triangle, if	(s) k = 5

Ans. (A) → s; (B) → p, q; (C) → r; (D) → p, q, s

Solution. The given lines are

L_1: x+3y -5 = 0

L_2 : 3x - ky - 1 = 0

L_3 :5x + 2y -12 = 0

(A) Three lines L_1, L_2,L_3 are concurrent if

$$\begin{vmatrix} 1 & 3 & 5 \\ 3 & -k & 1 \\ 5 & 2 & 12 \end{vmatrix} = 0 \Rightarrow 13k - 65 = 0 \Rightarrow k = 5$$

B) For $L_1 \parallel L_2 \Rightarrow \frac{1}{3} = \frac{-3}{k} \Rightarrow k = -9$

and $L_2 \parallel L_3 \Rightarrow \frac{3}{5} = \frac{-k}{2} \Rightarrow k = -\frac{6}{5}$

$\therefore$ (B) $\rightarrow$ (p), (q)

(C) Three lines L1, L2, L3 will form a triangle if no two of them are parallel and no three are concurrent

$\therefore$ $k \neq 5, -9, -6/5$

$\therefore$ (C) $\rightarrow$ r

(D) L1, L2,L3 do not form a triangle if either any two of these are parallel or the three are concurrent i.e. k = 5, – 9 , – 6/5

$\therefore$ (D) $\rightarrow$ (p), (q), (s)

Q.2. Match the Statements/Expressions in Column I with the Statements / Expressions in Column II and indicate your answer by darkening the appropriate bubbles in the 4 × 4 matrix given in the ORS.

Column I	Column II
(A) The minimum value of $\frac{x^2+2x+4}{x+2}$ is	**(p) 0**
(B) Let A and B be 3 × 3 matrices of real numbers, where A is symmetric, B is skew-symmetric, and (A + B) (A – B) = (A – B) (A + B). If (AB)t = (–1)k AB, where (AB)t is the transpose of the matrix AB, then the possible values of k are	**(q) 1**

(C) Let a = log3 log3 2. An integer k satisfying $1 < 2^{(-k+3^{-a})} < 2$, **must be less than** **(r) 2**

(D) If sin θ = cosφ , then the possible values of $\frac{1}{\pi}\left(\theta \pm \phi - \frac{\pi}{2}\right)$ **are** **(s) 3**

Ans. (A) → r; (B) → q, s; (C) → r, s; (D) → p, r

Solution.

(A) Let $y = \frac{x^2+2x+4}{x+2} \Rightarrow \frac{dy}{dx} = \frac{x^2+4x}{(x+2)^2} = 0$

$\Rightarrow \quad x = 0, -4$

$\frac{d^2y}{dx^2} = \frac{8}{(x+2)^3}$

At $x = 0$, $\frac{d^2y}{dx^2}$ is true

∴ y is min when x = 0, ∴ y min = 2

(B) As A is symmetric and B is skew symmetric matrix, we should have

$A^t = A$ and $B^t = -B$ ...(1)

Also given that (A + B) (A – B) = (A– B) (A + B)

$\Rightarrow A^2 - AB + BA - B^2 = A^2 + AB - BA - B^2$

⇒ 2BA = 2AB or AB = BA ...(2)

Now given that

$(AB)^t = (-1)^k AB$

$\Rightarrow (BA)^t = (-1)^k AB$ (using equation (2))

$\Rightarrow A^t B^t = (-1)$ kAB

$\Rightarrow -AB = (-1)^k AB$ [using equation(1)]

$\Rightarrow$ k should be an odd number

$\therefore$ (B) $\rightarrow$ (q), (s) (C)

Given that a = log3 log 3 2

$$\Rightarrow \log_3 2 = 3^a \Rightarrow \frac{1}{\log_2 3} = 3^a \text{ or } \log_2 3 = 3^{-a}$$

$$\Rightarrow \quad 3 = 2^{(3^{-a})} \quad \ldots(1)$$

$$\text{Now} \quad 1 < 2^{(-k+3^{-a})} < 2 \Rightarrow 1 < 2^{-k}.2^{3^{-a}} < 2$$

$$\Rightarrow 1 < 2^{-k}.3 < 2 \quad \text{(using eq (1))}$$

$$\Rightarrow \frac{1}{3} < 2^{-k} < \frac{2}{3} \Rightarrow \frac{3}{2} < 2^k < 3 \Rightarrow k = 1$$

$\therefore$k is less than 2 and 3

$\therefore$ (C) $\rightarrow$ (r), (s).

(D) $\sin\theta = \cos\phi \Rightarrow \cos\left(\frac{\pi}{2} - \theta\right) = \cos\phi$

$$\Rightarrow \frac{\pi}{2} - \theta = 2n\pi \pm \phi,\ n \in Z \Rightarrow \theta \pm \phi - \frac{\pi}{2} = -2n\pi$$

$$\Rightarrow \quad \frac{1}{\pi}\left(\theta \pm \phi - \frac{\pi}{2}\right) = -2n$$

$\therefore$ Here possible values of $\frac{1}{\pi}\left(\theta \pm \phi - \frac{\pi}{2}\right)$ are 0 and 2 for

n = 0, –1.

$\therefore$ D $\rightarrow$ (p) ,(r).

Integer Value Correct of Matrices and Determinants

Q. 1. Let w be the complex number $\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$. **Then the number of distinct complex numbers z satisfying** $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix}=0$ is equal to

Ans. 0

Solution.

We have $\omega=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}=\frac{-1+i\sqrt{3}}{2}$

$\therefore 1+\omega+\omega^2=0$ and $\omega^3=1$

Then $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix}=0$

$C_1 \leftrightarrow C_1+C_2+C_3$

$$\Rightarrow \begin{vmatrix} z+1+\omega+\omega^2 & \omega & \omega^2 \\ z+1+\omega+\omega^2 & z+\omega^2 & 1 \\ z+1+\omega+\omega^2 & 1 & z+\omega \end{vmatrix}=0$$

$$\Rightarrow z\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix}=0$$

$$\Rightarrow z\left[1\left(z^2+z\omega+z\omega^2+\omega^3-1\right)-\omega\left(z+\omega-1\right)+\omega^2\left(1-z-\omega^2\right)=0\right]$$

$$\Rightarrow z\left[z^2+z\omega+z\omega^2-z\omega-\omega^2+\omega+\omega^2-z\omega^2-\omega^4\right]=0$$

$$\Rightarrow z\left[z^2\right]=0\Rightarrow z^3=0\Rightarrow z=0$$

$\therefore$ z = 0 is the only solution.

Q. 2. Let k be a positive real n umber and

let $A = \begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}$

If det (adj A) + det (adj B) = 10^6. then [k] is equal to

[Note : adj M denotes the adjoint of square matrix M and [k] denotes the largest integer less than or equal k.

Ans. 4

Solution.

$$|A| = \begin{vmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 2k-1 & 0 & 2\sqrt{k} \\ 2\sqrt{k} & 1+2k & -2k \\ -2\sqrt{k} & 1+2k & -1 \end{vmatrix}, C_2 \to C_2 - C_3$$

$$= \begin{vmatrix} 2k-1 & 0 & 2\sqrt{k} \\ 4\sqrt{k} & 0 & 1-2k \\ -2\sqrt{k} & 1+2k & -1 \end{vmatrix}, R_2 \to R_2 - R_3$$

= (1 + 2k) (8k – 4k + $4k^2$+ 1) = (2k+ 1)3

Also B = 0 as B is skew symmetric of odd order..

∴ |Adj A| + |Adj B| = $|A|^2 + |B|^2$ = 106

⇒ $(2k + 1)^6 = 10^6$ ⇒ 2k + 1 = 10 ⇒ k= 4.5

∴ [k] = 4

Q. 3. Let M be a 3 × 3 matrix satisfying $M\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}-1\\2\\3\end{bmatrix}$, $M\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}1\\1\\-1\end{bmatrix}$, and $M\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\12\end{bmatrix}$. **Then the sum of the diagonal entries of M is**

Ans. 9

Solution.

$$\text{Let } M=\begin{bmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{bmatrix}$$

$$\text{then }\begin{bmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}-1\\2\\3\end{bmatrix}\Rightarrow \begin{matrix}b_1=-1\\ b_2=2\\ b_3=3\end{matrix}$$

$$\begin{bmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & c_3 & c_3\end{bmatrix}\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}1\\1\\-1\end{bmatrix}\Rightarrow \begin{matrix}a_1-b_1=1\\ a_2-b_2=1\\ a_3-b_3=-1\end{matrix}$$

$$\Rightarrow a_1=0, a_2=3, a_3=2$$

$$\text{and }\begin{bmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\12\end{bmatrix}\Rightarrow a_3+b_3+c_3=12 \;\Rightarrow c_3=7$$

∴ Sum of diagonal elements = $a_1 + b_2 + c_3 = 0 + 2 + 7 = 9$

Q. 4. The total number of distinct $x \in \mathbb{R}$ **for which** $\begin{vmatrix}x & x^2 & 1+x^3\\ 2x & 4x^2 & 1+8x^3\\ 3x & 9x^2 & 1+27x^3\end{vmatrix}=10$ is

Ans. 2

Solution.

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$$

$$\Rightarrow x^3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 3 & 9 & 1 \end{vmatrix} + x^6 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{vmatrix} = 10$$

Operating $C_2 - C_1$, $C_3 - C_1$ for both the determinants, we get

$$\Rightarrow x^3 \begin{vmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 3 & 6 & -2 \end{vmatrix} + x^6 \begin{vmatrix} 1 & 0 & 0 \\ 2 & 2 & 6 \\ 3 & 6 & 24 \end{vmatrix} = 10$$

$\Rightarrow x^3 (-4 + 6) + x^6 (48 - 36) = 10$

$\Rightarrow 2x^3 + 12x^6 = 10 \Rightarrow 6x^6 + x^3 - 5 = 0$

$$\Rightarrow (6x^3 - 5)(x^3 + 1) = 0 \Rightarrow x = \left(\frac{5}{6}\right)^{\frac{1}{3}}, -1$$

Q. 5. Let $z = \dfrac{-1+\sqrt{3}i}{2}$, where $i = \sqrt{-1}$, **and r, s ∈ {1, 2, 3}. Let** $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ **and I be the identity matrix of order 2. Then the total number of ordered pairs (r, s) for which $P^2 = -I$ is**

Ans. 1

Solution.

$$z = \frac{-1+i\sqrt{3}}{2} \Rightarrow z^3 = 1 \text{ and } 1 + z + z^2 = 0$$

$$P^2 = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}\begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$$

$$= \begin{bmatrix} z^{2r} + z^{4s} & z^{2s}\left((-z)^r + z^r\right) \\ z^{2s}\left((-z)^r + z^r\right) & z^{4s} + z^{2r} \end{bmatrix}$$

For $P^2 = -I$ we should have $z^{2r} + z^{4s} = -1$ and $z^{2s}((-z)^r + z^r) = 0$

$\Rightarrow z^{2r} + zs + 1 = 0$ and $(-z)^r + z^r = 0$

$\Rightarrow$ r is odd and s = r but not a multiple of 3.

Which is possible when s = r = 1

$\therefore$ only one pair is there.